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Magnetoelastic stability of a superconducting ring in its own field

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Summary.

The stability of the flexural vibrations of a superconducting ring in its own magnetic field is investigated. This problem is formulated as a perturbation problem: the final magnetic fields due to the deflected ring are considered as perturbations of the rigid-body fields. Both the rigid-body problem and the linearized perturbed problem are solved analytically. These solutions are expressed in Legendre functions. A so-called ring equation for the in-plane flexural vibrations of the slender ring is constructed. From this equation a frequency-current dispersion relation is derived. It turns out that the ring is stable against in-plane vibrations and that the eigenfrequency increases with increasing current.

1. Introduction

In certain modern technical equipments such as fusion reactors and MHD-devices large magnetic fields occur. These fields are generated by very high currents flowing in superconducting circuits. In the design of these devices the analysis of the vibrations and the stability of these superconducting circuits plays an important role.

The first research on this field is due to Leontovich and Shafranov, and to Dolbin and Morozov, [1]–[2], but the great impetus was given by Moon in cooperation with Chattopadhyay, [3]–[6]. An excellent survey of this field can be found in the book of F. Moon, [4], which contains a vast number of elaborated examples on the field of magnetoelastic stability. The above mentioned authors treated (among other problems) the problem of the stability of a flexible, conducting rod in its own magnetic field, and they showed — both theoretically and experimentally — that the straight configuration of the rod becomes unstable whenever the current exceeds a certain critical value. Moon [4] also investigated the stability of a circular coil in a transverse or toroidal external magnetic field ([4], Section 6.7.).

One particular problem not discussed in [4] concerns the stability of the flexural vibrations of a superconducting ring in its own magnetic field. This problem was solved by means of a variational method by Chattopadhyay in [6]. This approach resulted in a purely numerical analysis which obscured somewhat the underlying mechanics. Therefore, we have looked for an analytical treatment of the problem, in order to make it possible to indicate explicitly which are the effects that influence the stability of the ring.

In the present paper we investigate the stability of in-plane vibrations of a circular ring carrying an electric current which is confined on the surface of the ring (superconductivity). The investigation is based on a perturbation method; the final fields pertinent to the deflected ring are decomposed into the fields for the undeformed ring (rigid-body fields) and the perturbations on these fields, which are due to the deflections of the ring. Firstly, the rigid-body problem is solved. As a specific result the initial stresses due to the magnetic forces (Lorentz forces) in the undeformed coil are obtained (it turns out that these stresses play a dominant role in the ultimate stability criterion). Next, the linearized perturbed problem is solved, yielding expressions for the load of magnetic origin on the deformed ring. The total load on the deformed ring is now known, and from this an equation for the vibrational motion of the (slender) ring is derived. The solution of this equation leads to an expression for the eigenfrequencies of the conducting ring. This expression consists of two terms; one due to the initial stresses and the other due to the perturbations. The first term increases the frequency with increasing current (and, hence, has a stabilizing effect) whereas the second one causes the frequency to decrease. It turns out that the first term dominates the second one implying that the undeformed state of the ring is stable against vibrations in its own plane. This result is in correspondence with that of [6].

2. Basic equations

Consider a material body B placed in a vacuum. The body occupies *in its deformed configuration* a domain G in $\mathbf{R}^3$. The boundary of G is denoted by ∂G and $\mathbf{n}$ is the unit outward normal vector on ∂G. The Eulerian and Lagrangian coordinates of a material point P of B are $\mathbf{x}$ and $\mathbf{X}$, respectively.
To the body, which is assumed to be superconductive, an electric current I_0 is applied. For a superconductive body the current $\mathbf{J}$ is concentrated on the surface of the body ($\mathbf{J}$ is then the surface current density per unit of area). Moreover, the magnetic field inside the body is now zero. The magnetic field $\mathbf{B}$ in the vacuum outside the body has to satisfy

$$e_{ijk}B_{k,j}=0, \quad B_{i,i}=0, \quad \mathbf{B}\to\mathbf{0} \quad \text{as } |\mathbf{x}|\to\infty. \tag{2.1}$$

in the usual tensor notation, where $\;,i=\partial/\partial x_i$.
The external field $\mathbf{B}$ is related to the surface current $\mathbf{J}$ by the boundary conditions on ∂G, i.e. (μ_0 is the permeability in vacuum),

$$e_{ijk}B_j^+n_k=-\mu_0 J_i, \quad B_i^+n_i=J_in_i=0, \quad \text{for } \mathbf{x}\in\partial G, \tag{2.2}$$

where $^+$ indicates that the value of $\mathbf{B}$ must be taken at the boundary ∂G.
The Cauchy stresses T_{ij} have to satisfy the equations of motion (since there is no magnetic field inside the body, the body force is zero),

$$T_{ij,j}=\rho\ddot{U}_i, \tag{2.3}$$

where $\mathbf{U}=\mathbf{U}(\mathbf{x}, t)$ is the displacement vector and ρ the density. The stresses T_{ij} are related to the deformations by the common constitutive laws of elasticity (we tacitly assume here hyperelasticity, but we do not need the pertinent constitutive laws explicitly). The Lorentz force due to the current **J** produces a surface tension at ∂G according to

$$T_{ij}n_j = T_i = T_{ij}^{M+}n_j - T_{ij}^{M-}n_j$$

$$= \left[H_i^+ B_j^+ - \frac{\mu_0}{2}\delta_{ij}H_k^+ H_k^+\right]n_j = \tfrac{1}{2}e_{ijk}J_jB_k^+, \quad \text{for } \mathbf{x} \in \partial G. \tag{2.4}$$

Note. The above equations and boundary conditions are referred to the deformed configuration. In particular, the boundary conditions (2.2) and (2.4) hold on the deformed surface ∂G. Therefore the above system is highly nonlinear. In the next section a linearization procedure for this system will be presented.

3. Linearization procedure

The linearization procedure we shall present in this section is based, firstly, on the assumption that the general system of the preceding section has an equilibrium state and, secondly, on the condition that the final or x-state is a perturbation of this equilibrium state. We call the equilibrium solution the intermediate state and we denote its Euler coordinates by $\boldsymbol{\xi}$. The field quantities in this state are indicated by an upper index ^{0}. Our main purpose is to investigate the stability of this intermediate state. To this end, the non-linear equations of Section 2 must be linearized with respect to the $\boldsymbol{\xi}$-state.

For stability considerations, however, the deformations in the $\boldsymbol{\xi}$-state are not relevant. It is common practice to neglect the deformations and, thus, to identify the intermediate state with the rigid-body state. The electromagnetic fields in the rigid-body state are governed by the following set of equations and boundary conditions ((2.1)–(2.4), but now referred to the undeformed **X**-state):
in vacuum

$$e_{ijk}B_{k,j}^0 = 0, \quad B_{i,i}^0 = 0, \quad \mathbf{B}^0 \to \mathbf{0}, \quad \text{as } |\mathbf{X}| \to \infty, \tag{3.1}$$

at the surface of B

$$e_{ijk}B_j^{0+}N_k = -\mu_0 J_i^0, \quad B_i^{0+}N_i = 0, \quad J_i^0 N_i = 0, \tag{3.2}$$

where **N** is the unit outward normal vector at the undeformed surface and $,i$ must be read as $\partial/\partial X_i$ here.

The stresses in the intermediate state, the so called pre-stresses, T_{ij}^0 can, eventually, be determined from the equilibrium equations, the constitutive equations and the boundary conditions. However, in the sequel only the latter are needed. They read

$$T_{ij}^0 N_j = T_i^0 = \tfrac{1}{2}e_{ijk}J_j^0 B_k^{0+}. \tag{3.3}$$

As said before, the **x**-state must be considered as a perturbation of the $\boldsymbol{\xi}$-state. This perturbation is characterized by the displacement vector

$$\mathbf{u} = \mathbf{u}(\boldsymbol{\xi}, t) = \mathbf{x} - \boldsymbol{\xi}, \tag{3.4}$$

whose gradients are assumed to be small, i.e.

$$\|\text{grad}\,\mathbf{u}\| = \epsilon, \quad 0 < \epsilon \ll 1. \tag{3.5}$$

Note. As far as the electromagnetic part is concerned, we only consider quasi-static processes. Therefore, the time dependence of the perturbations is suppressed throughout this section.

Denoting the perturbations by lower case letters, we decompose the vacuum field into

$$\mathbf{B} = \mathbf{B}^0(\mathbf{x}) + \mathbf{b}(\mathbf{x}), \tag{3.6}$$

whereas the fields inside the body are decomposed according to

$$\mathbf{J} = \mathbf{J}^0(\boldsymbol{\xi}) + \mathbf{j}(\boldsymbol{\xi}) \tag{3.7}$$

and

$$T_{ij} = T_{ij}^0(\boldsymbol{\xi}) + t_{ij}(\boldsymbol{\xi}). \tag{3.8}$$

This separation of the fields allows a decomposition of the general set of nonlinear equations into a set for the rigid-body state (already given in (3.1)–(3.3)) and one for the perturbations. The latter will be linearized with respect to these perturbations. This results in the following set (since this linearization procedure is standard, cf. [7] or [8], we immediately give the results):
in vacuum

$$e_{ijk} b_{k,j} = 0, \quad b_{i,i} = 0, \quad \mathbf{b} \to \mathbf{0} \quad \text{as } |\boldsymbol{\xi}| \to \infty, \tag{3.9}$$

in the body *

$$t_{ij,j} - T_{ij,k} u_{k,j} = \rho \ddot{u}_i, \tag{3.10}$$

at the surface of B

$$e_{ijk} b_j^+ N_k + e_{ijk} B_{j,l}^+ u_l N_k - e_{ijk} B_j^+ u_{l,k} N_l = -\mu_0 j_i + \mu_0 J_i u_{j,k} N_j N_k, \tag{3.11}$$

$$b_i^+ N_i = B_i^+ u_{j,i} N_j - B_{i,j}^+ u_j N_i,$$

$$j_i N_i = J_i u_{j,i} N_j,$$

and

$$t_{ij} N_j = -T_i u_{l,k} N_l N_k + T_{ij} u_{k,j} N_k + t_i,$$

where

$$t_i = \tfrac{1}{2} e_{ijk} j_j B_k^+ + \tfrac{1}{2} e_{ijk} J_j \left(b_k^+ + B_{k,l}^+ u_l \right). \tag{3.12}$$

* Since from here on no confusion is possible, we omit the upper index ^{0}.

For hyperelastic materials the incremental stress relations are given by (cf. [9], Eq. (21.10))

$$t_{ij} = -T_{ij}u_{k,k} + T_{jk}u_{i,k} + T_{ik}u_{j,k} + \tau_{ij}, \tag{3.13}$$

where τ_{ij} is the incremental elastic stress which is directly related to the infinitesimal deformations by Hooke's law.

With (3.13), (3.10) and $(3.11)^4$ become

$$\tau_{ij,j} + f_i = \rho\ddot{u}_i, \quad f_i = \left(T_{jk}u_{i,k}\right)_{,j} \tag{3.14}$$

and

$$\tau_{ij}N_j = \tau_i = T_i u_{k,k} - T_k u_{i,k} - T_i u_{j,k}N_jN_k + t_i, \tag{3.15}$$

at the surface of B.

For later reference it is convenient to elaborate the above equations somewhat further. To this end, we use the relation

$$B_i^+ = B_j^+ N_j N_i + e_{ijk}e_{klm}B_l^+ N_m N_j = \mu_0 e_{ijk} J_j N_k, \tag{3.16}$$

according to (3.2). With (3.16) and $(3.2)^3$ the relation (3.3) reduces to

$$T_i = -\tfrac{1}{2}\mu_0(\mathbf{J}, \mathbf{J})N_i, \tag{3.17}$$

hence, $\mathbf{T}$ represents a purely normal stress. Furthermore, substitution of (3.16) into $(3.11)^1$ and $(3.11)^2$ yields

$$e_{ijk}\left(b_j^+ + B_{j,l}^+u_l\right)N_k = -\mu_0 j_i + \mu_0 J_k u_{j,k}N_iN_j, \tag{3.18}$$

and

$$\left(b_i^+ + B_{i,j}^+u_j\right)N_i = \mu_0 e_{ijk}J_j u_{l,i}N_kN_l =: \beta. \tag{3.19}$$

(the scalar β defined above turns out to be zero in the case of a ring; see Section 5). Analogous to the derivation of (3.16), we can deduce from (3.18) and (3.19)

$$b_i^+ + B_{i,j}^+u_j = \beta N_i + \mu_0 e_{ijk} j_j N_k. \tag{3.20}$$

With (3.16) and (3.19) the right-hand side of (3.12) can be expressed in $\mathbf{J}$ and $\mathbf{j}$. After some manipulations, in which also $(3.2)^3$ and $(3.11)^3$ are used, we arrive at

$$t_i = -\mu_0 j_j J_j N_i + \tfrac{1}{2}\mu_0 J_i J_j u_{k,j}N_k + \tfrac{1}{2}\beta e_{ijk}J_jN_k. \tag{3.21}$$

A useful result, which follows immediately from (3.21), reads

$$t_iN_i = -\mu_0 j_j J_j. \tag{3.22}$$

4. Configuration of the ring

Consider a slender ring, radius R, of circular cross-section, radius r_1 (see Fig. 1). A ring is called slender if $\delta \ll 1$, where

$$\delta = \frac{r_1}{R}. \tag{4.1}$$

The configuration of the ring is most easily described in toroidal coordinates. Starting from the cylindrical coordinates (r, ϕ, z) as shown in Fig. 1, we introduce toroidal coordinates (μ, η, ϕ) by (cf. [10], Section 10.3)

$$r = h \sinh \mu, \quad \phi = \phi, \quad z = h \sin \eta, \quad (\mu \geqslant 0, 0 \leqslant \eta < 2\pi, 0 \leqslant \phi < 2\pi), \tag{4.2}$$

where

$$h = h(\mu, \eta) = \frac{a}{\cosh \mu - \cos \eta}, \quad a = \sqrt{R^2 - r_1^2} = R\sqrt{1 - \delta^2}. \tag{4.3}$$

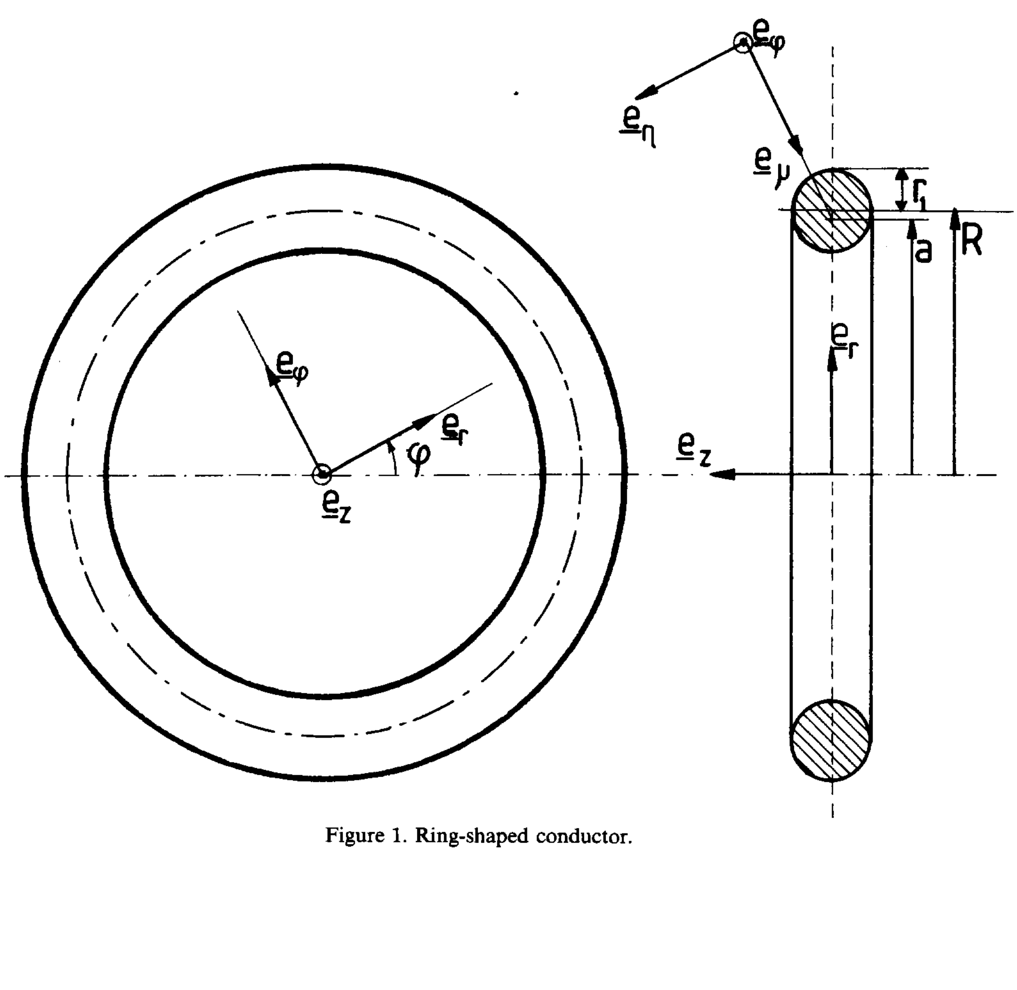

Figure 1. Ring-shaped conductor.

The surface of the ring, on which

$$\mathbf{N} = -\mathbf{e}_\mu, \tag{4.4}$$

is given by

$$\mu = \mu_1 := \log\left(\frac{R+a}{r_1}\right) = \log\left[\delta^{-1}\left(1+\sqrt{1-\delta^2}\right)\right]. \tag{4.5}$$

Then

$$\cosh\mu_1 = \frac{R}{r_1} = \delta^{-1}, \quad \sinh\mu_1 = \frac{a}{r_1} = \delta^{-1}\sqrt{1-\delta^2}. \tag{4.6}$$

Inside the ring one has $\mu_1 < \mu \leqslant \infty$, whereas the vacuum is given by $0 \leqslant \mu < \mu_1$, and μ, as well as η, tend to zero at infinity.

Let $\mathbf{u} = \mathbf{u}(r, \phi, z)$ be the displacement of the ring (in fact $\mathbf{u} = \mathbf{u}(r, \phi, z, t)$, but the time dependence is suppressed for the time being). Since we consider in-plane vibrations only, the displacement of the central line of the ring can be represented by

$$\mathbf{u}(R, \phi, 0) = w(\phi)\mathbf{e}_r + v(\phi)\mathbf{e}_\phi. \tag{4.7}$$

The ring is assumed inextensible. Then, the displacements are restricted by the condition

$$v' + w = 0, \quad \left({}' = \frac{d}{d\phi}\right). \tag{4.8}$$

For a slender ring (i.e. under the neglect of $O(\delta^2)$-terms) the displacement in an arbitrary point of the ring reads

$$u_r = w, \quad u_\phi = v - \frac{(r-R)}{R}(w' - v), \quad u_z = 0. \tag{4.9}$$

On account of (4.8), (4.9) implies

$$u_r|_\phi = -u_\phi|_r = \frac{w'-v}{R}, \quad u_\phi|_\phi = \frac{r-R}{rR}(v' - w''), \tag{4.10}$$

whereas all the remaining components of $u_i|_j$ (in cylindrical coordinates) are zero. The corresponding components of $\mathbf{u}$ in toroidal coordinates are (use Eq. (A.1) of Appendix A)

$$u_\mu = h(1 - \cosh\mu\cos\eta)\frac{w}{a}, \quad u_\eta = -h\sinh\mu\sin\eta\frac{w}{a}, \tag{4.11}$$

$$u_\phi = v - \left(\frac{h}{R}\sinh\mu - 1\right)(w' - v).$$

For later reference, we note that (see (A.3))

$$u_\mu|_\mu = u_\mu|_\eta = 0, \quad u_\mu|_\phi = -u_\phi|_\mu = \frac{h}{R}(1 - \cosh\mu\cos\eta)\frac{w'-v}{a}. \tag{4.12}$$

5. Rigid-body fields

For the solution of the system (3.1)–(3.2) in case of a ring or torus we refer to [11], Section 6. It was shown there that the rigid-body magnetic field **B** can be expressed in a vector potential

$$\mathbf{A} = A(\mu, \eta)\, \mathbf{e}_\phi,$$

by

$$\mathbf{B} = \text{rot}\, \mathbf{A}. \tag{5.1}$$

This immediately yields $B_\phi = 0$.

In [11], the following solution for $A(\mu, \eta)$ is obtained (cf. [11], Eq. (6.10)):

$$A(\mu, \eta) = \sqrt{\cosh \mu - \cos \eta}\, \sum_{n=0}^{\infty}{}' C_n P^1_{n-(1/2)}(\cosh \mu) \cos n\eta, \tag{5.2}$$

where

$$C_n = \frac{D Q^1_{n-(1/2)}(\cosh \mu_1)}{(4n^2 - 1) P^1_{n-(1/2)}(\cosh \mu_1)},$$

and the ′ in the summation sign indicates that the term with $n = 0$ has to be multiplied by $1/2$. Furthermore, $P^1_{n-(1/2)}$ and $Q^1_{n-(1/2)}$ are Legendre functions. The constant D follows from the condition for the total current and will be given below.

In accordance with (3.2), the corresponding current density is

$$\mathbf{J} = J(\eta) \mathbf{e}_\phi, \tag{5.3}$$

where (cf. [11], Eq. (6.11))

$$J(\eta) = -\frac{1}{\mu_0} B^+_\eta = -\frac{D}{4a\mu_0} \cdot \frac{(\cosh \mu_1 - \cos \eta)^{3/2}}{\sinh \mu_1} \sum_{n=0}^{\infty}{}' \frac{\cos n\eta}{p_n}, \tag{5.4}$$

with

$$p_n = P^1_{n-(1/2)}(\cosh \mu_1). \tag{5.5}$$

The coefficient D can now be determined from the condition for the total current

$$I_0 = \int_{\partial S} |\mathbf{J}|\, \mathrm{d}s = \int_0^{2\pi} \frac{J(\eta)}{h(\mu_1, \eta)}\, \mathrm{d}\eta, \tag{5.6}$$

which yields (cf. [11], Eq. (6.13))

$$D = -\frac{\mu_0 I_0}{\sqrt{2}} \sum_{n=0}^{\infty}{}' \frac{Q^1_{n-(1/2)}(\cosh \mu_1)}{(4n^2 - 1) P^1_{n-(1/2)}(\cosh \mu_1)}. \tag{5.7}$$

For a slender ring $\delta \ll 1$, and, hence, $\cosh \mu_1 = \delta^{-1}$ tends to infinity. With the use of the asymptotic expansions for the Legendre functions of large argument as given in Appendix B, one obtaines from (5.7)

$$D = \frac{2\sqrt{2}}{\pi^2}\mu_0 I_0 l\left(1 + O(\delta^2 \log \delta)\right), \tag{5.8}$$

where

$$l = \log\left(\frac{8}{\delta}\right) - 2. \tag{5.9}$$

Furthermore, $J(\eta)$ can be written as

$$\begin{aligned} J(\eta) &= -\frac{D}{4a\mu_0} \cdot \frac{\delta^{-1/2}(1 - \delta \cos \eta)^{3/2}}{(1 - \delta^2)^{1/2}} \cdot \frac{1}{2p_0}\left\{1 + \sum_{n=1}^{\infty} \frac{2p_0}{p_n} \cos n\eta\right\} \\ &= \frac{\hat{I}_0}{2\pi R}\delta^{-1}(1 - \delta \cos \eta)^{3/2} s(\eta), \end{aligned} \tag{5.10}$$

where we have introduced

$$\frac{\hat{I}_0}{2\pi R} = -\frac{D}{4a\mu_0(1 - \delta^2)^{1/2}} \cdot \frac{\delta^{1/2}}{2p_0} = \frac{I_0}{2\pi R}\left(1 + O(\delta^2 \log \delta)\right), \tag{5.11}$$

and

$$\begin{aligned} s(\eta) &= 1 + \sum_{n=1}^{\infty} \frac{2p_0}{p_n} \cos n\eta \\ &= 1 - \delta l \cos \eta - \tfrac{1}{4}\delta^2 l \cos 2\eta + O(\delta^3 \log \delta). \end{aligned} \tag{5.12}$$

The results obtained in this section thus far allow us the derivation of some relations for the stresses which are very useful in the sequel (see Section 7). Firstly, (3.17) reduces to

$$\mathbf{T} = \tfrac{1}{2}\mu_0 J^2(\eta)\mathbf{e}_\mu. \tag{5.13}$$

The equilibrium equation in r-direction in the intermediate state and under rotational symmetry reads

$$\frac{\partial(rT_{rr})}{\partial r} + r\frac{\partial T_{rz}}{\partial z} - T_{\phi\phi} = 0. \tag{5.14}$$

Integrating this equation over the cross-section S (with boundary ∂S) and applying Gauss' theorem, we obtain the relation

$$\int_S T_{\phi\phi}\, d\sigma = \int_{\partial S} rT_r\, ds, \tag{5.15}$$

which will be used later on (i.e in Section 7).

For later use, we also wish to calculate

$$\int_{\partial S} T_r \, \mathrm{d}s = \tfrac{1}{2}\mu_0 a \int_0^{2\pi} \frac{1 - \cosh \mu_1 \cos \eta}{(\cosh \mu_1 - \cos \eta)^2} J^2(\eta) \, \mathrm{d}\eta, \tag{5.16}$$

as follows from (5.13). With $J(\eta)$ according to (5.10)–(5.12) and for small δ, (5.16) becomes

$$\int_{\partial S} T_r \, \mathrm{d}s = \frac{\mu_0 I_0^2}{4\pi R}\left[\log\left(\frac{8}{\delta}\right) - \frac{1}{2}\right](1 + \mathrm{O}(\delta \log \delta)). \tag{5.17}$$

We, now, also evaluate (3.21). Firstly, we note that (3.19), (4.12) and (5.3) imply $\beta = 0$. Then (3.21) reduces to

$$t_\mu = \mu_0 J(\eta) j_\phi, \quad t_\eta = 0, \quad t_\phi = -\tfrac{1}{2}\mu_0 J^2(\eta) u_\mu |_\phi. \tag{5.18}$$

Finally, we note that (3.15) with (5.13), $(5.18)^3$ and (4.13) yields

$$\tau_\phi = 0. \tag{5.19}$$

6. Perturbed fields

The perturbed vacuum field $\mathbf{b}$ is completely determined by (3.9) together with the boundary condition $(3.11)^2$. Elaboration of $(3.11)^2$ for the case of the ring with use of the results of the preceding sections (i.e. $u_\mu|_\eta = B_\phi^+ = B_\mu^+ = 0$) leads to

$$b_\mu^+ = -B_\mu^+|_\mu u_\mu - B_\mu^+|_\eta u_\eta. \tag{6.1}$$

The right-hand side of (6.1) will now be expressed in $J(\eta)$. Since $\operatorname{div} \mathbf{B} = 0$ and $B_\phi^+ = B_\mu^+ = 0$, one has (consult Appendix A, Eq. (A.3), for the relevant formula for the covariant derivatives)

$$B_\mu^+|_\mu = -\frac{1}{h(\mu_1, \eta)} B_{\eta,\eta}^+ + \frac{1}{a} B_\eta^+ \sin \eta = \frac{\mu_0}{h(\mu_1, \eta)} J'(\eta) - \frac{\mu_0}{a} J(\eta) \sin \eta, \tag{6.2}$$

where, in the last step, (5.4) is used. On the other hand,

$$B_\mu^+|_\eta = \frac{1}{a} B_\eta^+ \sinh \mu_1 = -\frac{\mu_0}{a} J(\eta) \sinh \mu_1. \tag{6.3}$$

Substituting (6.2), (6.3) and (4.11) into (6.1), we obtain

$$b_\mu^+ = \frac{\mu_0 w(\phi)}{a}\left\{(\cosh \mu_1 \cos \eta - 1) J'(\eta) - (\cosh^2\mu_1 + \cosh \mu_1 \cos \eta - 2)\frac{J(\eta) \sin \eta}{\cosh \mu_1 - \cos \eta}\right\}. \tag{6.4}$$

With the use of $J(\eta)$ according to (5.10), the right-hand side of (6.4) can be expanded in powers of δ, yielding

$$b_\mu^+ = \frac{\mu_0 \hat{I}_0}{2\pi R} \frac{w(\phi)}{a} \delta^{-2}(1 - \delta \cos \eta)^{1/2} \Big\{ \big[-1 - \tfrac{5}{4} l \delta^2 + \tfrac{1}{2}\delta^2 \big] \sin \eta + \frac{\delta}{4}(1 + 4l) \sin 2\eta + O(\delta^3 \log \delta) \Big\}. \tag{6.5}$$

Since $\mathbf{b}$ is a rotational free-vector field (see $(3.9)^1$), it may be written as the gradient of a potential $\psi = \psi(\mu, \eta, \phi)$, according to

$$\mathbf{b} = \operatorname{grad} \psi. \tag{6.6}$$

Then $(3.9)^2$ implies

$$\Delta \psi = 0, \tag{6.7}$$

where Δ is the three-dimensional Laplace operator.

Led by the specific form of the boundary condition (6.5), we introduce the following separation of variables for ψ,

$$\psi(\mu, \eta, \phi) = w(\phi)(\cosh \mu - \cos \eta)^{1/2} \sum_{n=1}^{\infty} G_n(\mu) \sin n\eta. \tag{6.8}$$

In order that this separation is consistent with (6.7), $w(\phi)$ has to satisfy

$$w''(\phi) + m^2 w(\phi) = 0, \quad m \in N. \tag{6.9}$$

Since we are only interested in the lowest buckling value, we may take for $w(\phi)$ the first bending mode of the ring, i.e.

$$w(\phi) = W \cos 2\phi, \quad (\text{or } m = 2). \tag{6.10}$$

With (6.8) and (6.10), (6.7) reduces to an ordinary differential equation for $G_n(\mu)$,

$$\frac{1}{\sinh \mu} \frac{\mathrm{d}}{\mathrm{d}\mu}\left(\sinh \mu \frac{\mathrm{d}G_n}{\mathrm{d}\mu} \right) - \frac{4 G_n}{\sinh^2 \mu} - \left(n^2 - \tfrac{1}{4} \right) G_n = 0, \tag{6.11}$$

holding for $0 \leqslant \mu < \mu_1$. In this region the general solution of (6.11) constitutes of the so-called ring functions $P^2_{n-(1/2)}(\cosh \mu)$ (cf. [12], Sec. 3.9.2).

Hence, the general solution of (6.7) can be written as

$$\psi = -\frac{D w(\phi)}{4a} (\cosh \mu - \cos \eta)^{1/2} \sum_{n=1}^{\infty} d_n \frac{p_n^{(2)}(\mu)}{p_n^{(2)}(\mu_1)} \sin n\eta. \tag{6.12}$$

where

$$p_n^{(2)}(\mu) = P^2_{n-(1/2)}(\cosh \mu). \tag{6.13}$$

Elaboration of (6.6) at $\mu = \mu_1$ with the aid of (6.12) and (A.2), yields

$$b_\mu^+ = \frac{\cosh \mu_1 - \cos \eta}{a} \left.\frac{\partial \psi}{\partial \mu}\right|_{\mu_1}$$

$$= -\frac{Dw(\phi)}{4a^2}(\cosh \mu_1 - \cos \eta)^{1/2} \frac{\sinh \mu_1}{2}$$

$$\times \sum_{n=1}^{\infty} \{1 + 2\delta(\cosh \mu_1 - \cos \eta)\Pi_n\} \; d_n \sin n\eta, \qquad (6.14)$$

where Π_n is introduced by means of the relation

$$\left[\frac{\mathrm{d}}{\mathrm{d}\mu}\left(\frac{p_n^{(2)}(\mu)}{p_n^{(2)}(\mu_1)}\right)\right]_{\mu=\mu_1} = \delta \Pi_n \sinh \mu_1, \qquad (6.15)$$

so

$$\Pi_n = \frac{1}{\delta}\left[\frac{\mathrm{d}}{\mathrm{d}z}\left\{\log | P^2_{n-(1/2)}(z) |\right\}\right]_{z=\delta^{-1}}. \qquad (6.16)$$

As can be deduced from the asymptotic expansions listed in Appendix B,

$$\Pi_1 = \tfrac{1}{2} + \tfrac{15}{4}\delta^2 l - 2\delta^2 + \mathrm{O}(\delta^4 \log \delta), \qquad (6.17)$$

$$\Pi_2 = \tfrac{3}{2} + \mathrm{O}(\delta^2 \log \delta),$$

$$\Pi_3 = \tfrac{5}{2} + \mathrm{O}(\delta^2 \log \delta),$$

for small δ. For convenience, we replace the coefficients d_n in (6.12) by

$$\hat{d}_n = 2p_0(1-\delta^2)^{1/2}\delta^{-(n-1)}d_n. \qquad (6.18)$$

Then (6.12) and (6.14) become

$$\psi = \mu_0 w(\phi) \frac{\hat{I}_0}{2\pi R}(\cosh \mu - \cos \eta)^{1/2} \sum_{n=1}^{\infty} \delta^{(n-(3/2))} \hat{d}_n \frac{p_n^{(2)}(\mu)}{p_n^{(2)}(\mu_1)} \sin n\eta, \qquad (6.19)$$

and

$$b_\mu^+ = \frac{\mu_0 w(\phi)}{a} \frac{\hat{I}_0}{2\pi R} \sinh \mu_1 (\cosh \mu_1 - \cos \eta)^{1/2}$$

$$\cdot \sum_{n=1}^{\infty} \{\tfrac{1}{2} + \Pi_n - \delta \Pi_n \cos \eta\} \delta^{(n-(3/2))} \hat{d}_n \sin n\eta, \qquad (6.20)$$

where we have used (5.11) for D.

As further analysis will show, the coefficients $\hat{d}_n$ are O(1) for $\delta \to 0$. Then, it is possible to expand the right-hand side of (6.20) in powers of δ, which with the use of (6.17) leads to

$$b_\mu^+ = \frac{\mu_0 \hat{I}_0}{2\pi R}\frac{w}{a}\delta^{-2}(1-\delta\cos\eta)^{1/2}\{[(1+\tfrac{15}{4}\delta^2 l - \tfrac{5}{2}\delta^2)\hat{d}_1 - \tfrac{3}{4}\delta^2\hat{d}_2]\sin\eta + \delta[-\tfrac{1}{4}\hat{d}_1 + 2\hat{d}_2]\sin 2\eta + \delta^2[-\tfrac{3}{4}\hat{d}_2 + 3\hat{d}_3]\sin 3\eta + O(\delta^3\log\delta)\}. \tag{6.21}$$

Equating (6.5) and (6.21), we obtain

$$\hat{d}_1 = -1 + \tfrac{23}{8}\delta^2 l - 2\delta^2 + O(\delta^4\log\delta), \tag{6.22}$$

$$\hat{d}_2 = \tfrac{1}{2}l + O(\delta^2\log\delta),$$

$$\hat{d}_3 = \tfrac{1}{8}l + O(\delta^2\log\delta).$$

We conclude this section by giving the solution for the disturbed current density **j**. Starting from the boundary conditions (3.11)[1] and (3.11)[3], and using (4.12), (5,3) and $B_\mu^+ = B_\phi^+ = 0$, we arrive at

$$j_\mu = J(\eta)u_\mu|_\phi, \quad j_\eta = \frac{1}{\mu_0}b_\phi^+ + \frac{1}{\mu_0}B_\phi^+|_i u_i, \tag{6.23}$$

$$j_\phi = -\frac{1}{\mu_0}b_\eta^+ - \frac{1}{\mu_0}B_\eta^+|_i u_i.$$

In the sequel only an explicit expression for j_ϕ is needed. Evaluation of (6.23) yields (realize that $B_\eta|_\mu = B_\mu|_\eta$)

$$j_\phi = -\frac{1}{\mu_0}b_\eta^+ - \frac{1}{\mu_0}B_\mu^+|_\eta u_\mu - \frac{1}{\mu_0}B_\eta^+|_\eta u_\eta$$

$$= -\frac{1}{\mu_0}\frac{\cosh\mu_1 - \cos\eta}{a}\frac{\partial\psi}{\partial\eta}(\mu=\mu_1) - \frac{1}{\mu_0}\frac{\sinh\mu_1}{a}B_\eta^+ u_\mu$$

$$-\frac{1}{\mu_0}\frac{\cosh\mu_1 - \cos\eta}{a}B_{\eta,\eta}^+ u_\eta, \tag{6.24}$$

which after an expansion in powers of δ results in

$$j_\phi = -\frac{I_0}{2\pi R}\frac{w(\phi)}{R}(3l - \tfrac{5}{2})\cos\eta(1 + O(\delta\log\delta)). \tag{6.25}$$

7. Equation of motion

A global equation of motion for the in-plane vibrations of a ring can be obtained by an integration of the local equations of motion (3.14) over the cross-section of the ring under the use of the boundary conditions (3.15). This derivation is straightforward and, therefore, will not be repeated here. The resulting equations can already be found in Love [13], but we prefer here to refer to [4], Eqs. (6–7.3) and (6–7.4). For in-plane vibrations, we set $u_y = 0$, and then also $N_y = G_x = H = 0$. After elimination of N_x and T from $(6\text{–}7.4)^2$ by means of $(6\text{–}7.3)^1$ and $(6\text{–}7.3)^3$, and with the aid of the well-known constitutive equation for the bending moment in a slender inextensible ring (cf. [4], (6–7.6)),

$$G_y = -\frac{EI}{R^2}(w'' + w), \tag{7.1}$$

one obtains (here we use $-w$ and v instead of u_x and u_z as in [4], respectively)

$$w^{\mathrm{v}} + 2w''' + w' = \frac{R^2}{EI}\{\Gamma - \rho A R^2(\ddot{w}' + \ddot{v})\}. \tag{7.2}$$

Here, E is Young's modulus and

$$I = \frac{\pi}{4} r_1^4, \quad A = \pi r_1^2,$$

are the moment of inertia and the area of the cross-section, respectively. Moreover, Γ is a load parameter, which arises from the body force $\mathbf{f}$ (in (3.14)) and the surface tension τ (in (3.15)) according to

$$\Gamma = \Gamma(\phi) = RK_r'(\phi) + RK_\phi(\phi) + L''(\phi) + L(\phi), \tag{7.3}$$

where

$$K_r(\phi) = \int_S r f_r \, \mathrm{d}\sigma + \int_{\partial S} r\tau_r \, \mathrm{d}s, \tag{7.4}$$

$$K_\phi(\phi) = \int_S r f_\phi \, \mathrm{d}\sigma + \int_{\partial S} r\tau_\phi \, \mathrm{d}s, \tag{7.5}$$

and

$$L(\phi) = \int_S (r - R) r f_\phi \, \mathrm{d}\sigma + \int_{\partial S} (r - R) r\tau_\phi \, \mathrm{d}s. \tag{7.6}$$

Hence, K_r and K_ϕ are forces in r- and ϕ-direction, respectively, and L is a couple about the z-axis.

According to $(3.14)^2$ we can write

$$f_i = S_{ij,j}, \quad \text{with } S_{ij} = T_{jk} u_{i,k}, \tag{7.7}$$

or, in components, with the aid of (4.10) and of $T_{r\phi} = T_{z\phi} = 0$,

$$S_{r\phi} = u_r|_\phi T_{\phi\phi}, \quad S_{\phi r} = u_\phi|_r T_{rr}, \quad S_{\phi\phi} = u_\phi|_\phi T_{\phi\phi}, \quad S_{\phi z} = u_\phi|_r T_{zr}, \tag{7.8}$$

$$S_{rr} = S_{rz} = S_{zr} = S_{z\phi} = S_{zz} = 0.$$

This implies

$$f_r = \frac{1}{r}S_{r\phi,\phi} - \frac{1}{r}S_{\phi\phi}, \quad f_\phi = \frac{1}{r}S_{\phi\phi,\phi}, \tag{7.9}$$

in which, for the second relation, (5.14) is used. Hence, with $\mu_r|_\phi$ according to (4.11),

$$rf_{r,\phi} + rf_\phi = S_{r\phi,\phi\phi} = \frac{w''' - v''}{R}T_{\phi\phi}. \tag{7.10}$$

Furthermore, (3.15) with $u_\mu|_\mu = 0$ and $T_\phi = 0$ and with (4.10) gives

$$\tau_r = \frac{r-R}{rR}(v' - w'')T_r + t_r, \tag{7.11}$$

while $\tau_\phi = 0$ according to (5.19).

Use of the preceding results in (7.4) and (7.5) yields

$$\begin{aligned} RK_r'(\phi) + RK_\phi(\phi) &= (w''' - v'')\int_S T_{\phi\phi}\,\mathrm{d}\sigma - (w''' - v'')\int_{\partial S}(r-R)T_r\,\mathrm{d}s \\ &\quad + R\frac{\mathrm{d}}{\mathrm{d}\phi}\int_{\partial S} rt_r\,\mathrm{d}s \\ &= (w''' - v'')R\int_{\partial S} T_r\,\mathrm{d}s + R\frac{d}{d\phi}\int_{\partial S} rt_r\,\mathrm{d}s, \end{aligned} \tag{7.12}$$

on account of (5.15).

Finally we note that

$$L(\phi) = \int_S (r-R)rf_\phi\,\mathrm{d}\sigma = -(w''' - v'')\int_S \frac{(r-R)^2}{rR}T_{\phi\phi}\,\mathrm{d}\sigma, \tag{7.13}$$

which is small of $O(\delta^2)$ compared with the first term on the right-hand side of (7.12). For a slender ring this contribution may be neglected, so

$$L(\phi) = 0. \tag{7.14}$$

Substituting (7.12) and (7.14) into (7.3) we arrive at

$$\Gamma(\phi) = (w''' - v'')R\int_{\partial S} T_r\,\mathrm{d}s + R\frac{d}{d\phi}\int_{\partial S} rt_r\,\mathrm{d}s, \tag{7.15}$$

which with (4.8) and (6.10) ultimately becomes

$$\Gamma(\phi) = -3w'(\phi)R\int_{\partial S} T_r \, \mathrm{d}s + R\frac{d}{d\phi}\int_{\partial S} rt_r \, \mathrm{d}s. \tag{7.16}$$

This result shows us that the force on the ring arises from two sources. The first is due to the pre-stresses (i.e. pertinent to the rigid-body fields). This contribution results in a circumferential tension in the ring, and, hence, has a stabilizing effect. The second contribution to Γ is due to the perturbed fields. As the further analysis will show, this term has a destabilizing effect on the vibrations of the ring.

The first term on the right-hand side of (7.16) is already calculated in Section 5 (see (5.17)). For an evaluation of the second term, we start from (5.18)[1,2], which imply

$$t_r = \frac{1-\cosh\mu_1\cos\eta}{\cosh\mu_1-\cos\eta}t_\mu = \mu_0 J(\eta)\frac{1-\cosh\mu_1\cos\eta}{\cosh\mu_1-\cos\eta}j_\phi, \tag{7.17}$$

and which with (5.10) and (6.25) can be worked out for small δ into

$$t_r = \frac{\mu_0 I_0^2}{4\pi^2 R^2}\frac{w(\phi)}{R}\delta^{-1}(3l-\tfrac{5}{2})\cos^2\eta(1+\mathrm{O}(\delta\log\delta)). \tag{7.18}$$

Hence, with (5.9),

$$\int_{\partial S} rt_r \, \mathrm{d}s = \frac{3\mu_0 I_0^2}{4\pi R}w(\phi)\left[\log\left(\frac{8}{\delta}\right)-\frac{17}{6}\right](1+\mathrm{O}(\delta\log\delta)). \tag{7.19}$$

Substitution of (5.17) and (7.19) into (7.16) finally results in

$$\Gamma(\phi) = -\frac{7}{4\pi}\mu_0 I_0^2 w'(\phi)(1+\mathrm{O}(\delta\log\delta)), \tag{7.20}$$

or, with $w(\phi)$ according to (6.10),

$$\Gamma(\phi) = \frac{7}{2\pi}\mu_0 I_0^2 W\sin 2\phi(1+\mathrm{O}(\delta\log\delta)). \tag{7.21}$$

8. Frequency-current dispersion relation

For the vibrating ring, we have to consider $w = w(\phi, t)$, instead of $w = w(\phi)$, and, therefore, we have to replace (6.10) by

$$w(\phi, t) = W(t)\cos 2\phi = \hat{W}e^{i\omega t}\cos 2\phi, \tag{8.1}$$

where ω is the frequency of the vibration. Then, according to (4.8),

$$v(\phi, t) = -\tfrac{1}{2}\hat{W}e^{i\omega t}\sin 2\phi. \tag{8.2}$$

Substitution of (8.1), (8.2) and (7.21) into the equation of motion (7.2) yields the dispersion equation

$$\hat{W}e^{i\omega t}\sin 2\phi\left\{-18-\frac{7R^2}{2\pi EI}\mu_0^2 I_0+\frac{5}{2}\frac{\rho A R^4}{EI}\omega^2\right\}=0. \tag{8.3}$$

With the definition of

$$\omega_0=\left(\frac{36EI}{5\rho A R^4}\right)^{1/2}, \tag{8.4}$$

the eigenfrequency of the free ring, the following relation for the frequency ω is obtained from (8.3)

$$\omega^2=\omega_0^2\left(1+\frac{7R^2}{36\pi EI}\mu_0 I_0^2\right). \tag{8.5}$$

For a conservative problem (as is the one considered here) instability occurs when ω becomes zero. Since the right-hand side of (8.5) remains positive for all values of I_0, we, thus, conclude that the equilibrium state of the ring is always stable against vibrations in its plane. Furthermore, it is seen from (8.5) that the current has the tendency to increase the eigenfrequency of the ring.

9. Conclusions

In the preceding analysis it is shown that the natural state of a superconducting ring-shaped coil is stable against in-plane vibrations, and it turned out that the eigenfrequency of the current-carrying ring increases with increasing current. The current term in the dispersion relation (8.5) is made up of two contributions: one originating from the initial magnetic field of the undeformed ring and the other from the perturbed self field due to the deflections of the ring. In order to investigate in detail the influences of these two contributions, we reconsider the results (7.16), (5.17) and (7.19), the first of which can be written as

$$\Gamma(\phi)=\Gamma_1(\phi)+\Gamma_2(\phi). \tag{9.1}$$

Here, $\Gamma_1(\phi)$ is the contribution due to the initial magnetic field and is equal to (see (7.16) and (5.17), with $w'(\phi)=-2w(\phi)$),

$$\Gamma_1(\phi)=\frac{3\mu_0 I_0^2}{2\pi}\left[\log\left(\frac{8}{\delta}\right)-\frac{1}{2}\right]w(\phi), \tag{9.2}$$

whereas $\Gamma_2(\phi)$ is due to the perturbed field and equals (see (7.19))

$$\Gamma_2(\phi)=\frac{3\mu_0 I_0^2}{2\pi}\left[-\log\left(\frac{8}{\delta}\right)+\frac{17}{6}\right]w(\phi). \tag{9.3}$$

The undeformed configuration of the ring is stable if $\Gamma(\phi)$ is positive for $w > 0$. We therefore may state that Γ_1 (i.e. the initial magnetic field) has a stabilizing effect, while Γ_2 (i.e. the perturbed field) has a destabilizing effect. However, this only holds as far as these terms are considered separately. When taken together, the main terms in Γ_1 and Γ_2 (i.e. the terms of $O(\log \delta)$) cancel each other. This implies that the (in)stability of the ring is in fact determined by the second terms in the expressions for Γ_1 and Γ_2 (which are of O(1) for small δ) and out of these two terms the contribution of Γ_2 is positive and exceeds the one of Γ_1, thus leading to the stability of the ring. Hence, we conclude that the crucial terms as far as the stability concerns are not the first, main-order, terms in Γ_1 and Γ_2, but it just are the second terms in these expressions. To get an exact numerical value for these terms, the precise calculations as presented in this paper are needed.

In order to illustrate more explicitly the influence of the current I_0 on the eigenfrequency ω, let us use the following numerical values (these values are according to [6])

$$E = 8 \times 10^{10} \text{ N/m}^2, \quad I = 2.2 \times 10^{-4} \text{ m}^4,$$
$$R = 3.03 \text{ m}, \quad \mu_0 = 4\pi \times 10^{-7} \text{ H/m}.$$

For these values the dispersion relation (8.5) becomes

$$\omega^2 = \omega_0^2 (1 + 4.06 \times 10^{-14} I_0^2). \tag{9.4}$$

This result agrees, at least in order of magnitude, reasonably well with that of Chattopadhyay, [6], Eq. (42). In this aspect it must be mentioned that in [6] it was assumed that the current is uniformly distributed over the cross-section.

In this paper, only the in-plane vibrations are considered. The out-of-plane vibrations can be treated in an analogous way. However, and this is confirmed by the results of [6], there seems to be no reason to expect that this solution will yield essentially different results. Also, the correspondence between our results and those of [6], suggests that the specific distribution of the current over the cross-section (e.g. a uniform current instead of a surface current) is not of essential influence. A different current-distribution will change the numerical value of the coefficient of I_0^2 in the dispersion relation, but it is not to be expected that it can disturb the stability of the ring. However, a complete justification of this expectation can only be provided by the explicit solutions of the underlying problems.

Appendix A

Between the unit vectors of a set of cylindrical and one of toroidal coordinates (see eq. (4.7)) the following relations exist

$$\begin{aligned} \boldsymbol{e}_r &= \frac{h}{a}(1 - \cosh\mu \cos\eta)\boldsymbol{e}_\mu - \frac{h}{a}\sinh\mu \sin\eta \boldsymbol{e}_\eta, \\ \boldsymbol{e}_\phi &= \boldsymbol{e}_\phi, \\ \boldsymbol{e}_z &= -\frac{h}{a}\sinh\mu \sin\eta \boldsymbol{e}_\mu - \frac{h}{a}(1 - \cosh\mu \cos\eta)\boldsymbol{e}_\eta. \end{aligned} \tag{A.1}$$

From these relations, the following expressions for the covariant derivatives of an arbitrary scalar field $\psi=\psi(\mu, \eta, \phi)$ or vector field $\mathbf{v}=\mathbf{v}(\mu, \eta, \phi)$ can be derived ($|_i$ denotes covariant differentiation)

$$\psi|_\mu=\frac{1}{h}\psi_{,\mu}, \quad \psi|_\eta=\frac{1}{h}\psi_{,\eta}, \quad \psi|_\phi=\frac{1}{h \sinh \mu}\psi_{,\phi}, \tag{A.2}$$

and

$$v_\mu|_\mu=\frac{1}{h}\left(v_{\mu,\mu}-\frac{h}{a}v_\eta \sin \eta\right), \quad v_\eta|_\mu=\frac{1}{h}\left(v_{\eta,\mu}+\frac{h}{a}v_\mu \sin \eta\right), \quad v_\phi|_\mu=\frac{1}{h}v_{\phi,\mu},$$

$$v_\mu|_\eta=\frac{1}{h}\left(v_{\mu,\eta}+\frac{h}{a}v_\eta \sinh \mu\right), \quad v_\eta|_\eta=\frac{1}{h}\left(v_{\eta,\eta}-\frac{h}{a}v_\mu \sinh \mu\right), \quad v_\phi|_\eta=\frac{1}{h}v_{\phi,\eta},$$

$$v_\mu|_\phi=\frac{1}{h \sinh \mu}\left(v_{\mu,\phi}-\frac{h}{a}(1-\cos \eta \cosh \mu)v_\phi\right),$$

$$v_\eta|_\phi=\frac{1}{h \sinh \mu}\left(v_{\eta,\phi}+\frac{h}{a}v_\phi \sinh \mu \sin \eta\right),$$

$$v_\phi|_\phi=\frac{1}{h \sinh \mu}\left(v_{\phi,\phi}+\frac{h}{a}(1-\cosh \mu \cos \eta)v_\mu-\frac{h}{a}v_\eta \sinh \mu \sin \eta\right). \tag{A.3}$$

Appendix B

For Legendre functions of large argument the following asymptotic expansions hold:

$$P^1_{-1/2}(z)=-\frac{1}{\pi\sqrt{2}}z^{-1/2}(\log 8z-2)\left\{1+\tfrac{7}{16}z^{-2}-\frac{1}{16}\frac{z^{-2}}{\log 8z-2}+\mathrm{O}(z^{-4})\right\},$$

$$P^1_{1/2}(z)=\frac{\sqrt{2}}{\pi}z^{1/2}(1+\mathrm{O}(z^{-2}\log z)), \quad P^1_{3/2}(z)=\frac{4\sqrt{2}}{\pi}z^{3/2}(1+\mathrm{O}(z^{-2})),$$

$$P^2_{1/2}(z)=-\frac{z^{1/2}}{\pi\sqrt{2}}(1+\tfrac{61}{16}z^{-2}-\tfrac{15}{8}z^{-2}\log 8z+\mathrm{O}(z^{-7/2})),$$

$$P^2_{3/2}(z)=\frac{2\sqrt{2}}{\pi}z^{3/2}(1+\mathrm{O}(z^{-2})), \quad P^2_{5/2}(z)=\frac{16\sqrt{2}}{\pi}z^{5/2}(1+\mathrm{O}(z^{-2})),$$

$$P^m_{n-1/2}(z)=\mathrm{O}(z^{n-1/2}), \quad (n\in\mathbb{N}),$$

$$Q^1_{-1/2}(z)=-\frac{\pi}{2\sqrt{2}}z^{-1/2}(1+\mathrm{O}(z^{-2})), \quad Q^m_{n-1/2}(z)=\mathrm{O}(z^{-n-1/2}). \tag{B.1}$$

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